Heuristic Solution of the Multisource Weber Problem as a p-Median Problem

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Abstract. Good heuristic solutions for large Multisource Weber problems can be obtained by solving related p-median problems in which potential locations of the facilities are users locations and then solving Weber problems for the sets of users of each facility.

Keywords. Location; Continuous; Heuristic.

Weber's problem is to locate a facility in the Euclidean plane in order to minimize the sum of its (weighted) distances to the locations of a given set of users. It is central to continuous location theory, and has been extended in many ways, see Wesolowsky (1993) for a recent survey. Weiszfeld's (1937) iterative algorithm, extended to ensure convergence even if an iterate coincides with a user's location (e.g. Brimberg and Love (1993)), is the standard way to solve Weber's problem. Rosen and Xue (1991) suggest accelerations. Considering simultaneous location of several facilities leads to two generalizations: (i) the *Multifacility Weber problem* in which several facilities each producing a different product (or rendering a different service) are to be located in order to minimize the sum of weighted distances between all facilities and all users as well as between the facilities, and (ii) the *Multisource Weber problem* in which several facilities producing the same product are to be located in order to minimize the sum of weighted distances from all users to their closest facility. While the former problem is one of convex optimization for which practically efficient algorithms (e.g. Calamai and Conn (1980), Overton (1983)) and a polynomial one (Xue, Rosen and Pardalos (1996)) are available, the later one is NP-hard (Megiddo and Supowit (1984)). Exact algorithms for its solution are based on branch-and-bound (Kuenne and Soland (1972), Ostresh (1973-b), (1973-c), partitioning the users space (Ostresh (1973-a), (1975), Rosing (1992)), d.-c. programming (Chen, Hansen, Jaumard and Tuy (1992)) and column generation (Hansen, Jaumard, Krau and du Merle (1997)). Except for the case of two facilities, which can be solved with 10000 users (Chen, et. al. (1992)), large problems, are hard to solve exactly. Starting with Cooper's work (Cooper (1963), (1964)) of which the

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alternate location - allocation algorithm (ALT) is best known, many heuristics for the Multisource Weber problem have been proposed, see e.g. Chen *et al.* (1992) for a short survey and discussion. Recently, Bongartz, Calamai and Conn (1994) proposed a projection method (MPROJ) for the Multisource Weber problem (with any ℓ_r norm for distance). This method relaxes the {0,1} constraints on the allocations, and solves the problem in $\mathbb{R}^{2n+pn-p}$ space, where n and p denote the numbers of users and of facilities respectively. Exploiting the specific structure of the problem obtained in that way, simple projection formulas on subspaces of a domain are derived (instead of solving the system of equations in general), and used in finding descent directions (in the framework of active set methods for nonlinear programming). MPROJ is a stable algorithm that iterates until convergence is obtained. Under mild conditions, it finds local minima with integer allocation variables and exibits quadratic convergence. Descending a given number of times from randomly generated solutions, or from solutions obtained by partitioning users in sets of succesive ones along a traveling salesman tour, provides a good solution. Bongartz *et al.* (1994) compare MPROJ with four heuristics from the literature. As it outperforms them, it may be considered state-of-the-art.

The purpose of the present paper is (i) to revive and make operational a forgotten heuristic of Cooper (1963) for the Multisource Weber problem, and (ii) to compare it experimentally with MPROJ and with the multistart alternate location-allocation heuristic (MALT). To this effect, the similarity between the Multisource Weber problem and its discrete counterpart, i.e., the p-median problem, is exploited. The Multisource Weber problem can be formulated as follows (Kuenne and Soland (1972)):

$$\min_{x_i,y_i,z_{ij}} \ \sum_{i=1}^{p} \sum_{j=1}^{n} z_{ij} \cdot w_j \cdot d_j(x_i,y_i)$$

subject to
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (MW)

$$\sum_{i=1}^{p} z_{ij} = 1, \quad j = 1,2,\ldots,n,$$

$$z_{ij} \in [0,1] \quad i = 1,2,\ldots,p; \ j = 1,2,\ldots,n,$$

where p facilities must be located to satisfy the demand of n users, x_i, y_i denote the coordinates of the ith facility, d_j the Euclidean distance from (x_i,y_i) to the jth user, w_j the demand (or weight) of the jth user and z_{ij} the fraction of this demand which is satisfied from the ith facility. It is easy to see that there is always an optimal solution with all $z_{ij} \in \{0,1\}$, i.e., each user satisfied from a single facility.

In the p-Median problem facilities must be located among a given set of sites. Otherwise the problem is the same as MW. Using similar notation to above the p-Median problem can be written

$$\min_{t_i,z_{ij}} \ \sum_{i=1}^{q} \sum_{j=1}^{n} z_{ij} \cdot w_j \cdot d_{ij}$$

subject to $\hspace{11cm}$ (PM)

$$\sum_{i=1}^{q} z_{ij} = 1, \quad j = 1, 2, \ldots, n,$$

$$z_{ij} \leq t_i, \quad i = 1, 2, \ldots, q; \ j = 1, 2, \ldots, n,$$

$$\sum_{i=1}^{q} t_i = p,$$

$$t_i \in \{0, 1\} \quad i = 1, 2, \ldots, q,$$

$$z_{ij} \in \{0, 1\} \quad i = 1, 2, \ldots, q; \ j = 1, 2, \ldots, n.$$

where $q > p$ sites are considered for location of p facilities, $t_i = 1$ if a facility is located at site i, $t_i = 0$ otherwise and d_{ij} denotes the distance between site i and user j.

p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	20	25	40	45	50
#	1	2	4	4	5	5	6	7	7	3	6	7	9	11	15	19	30	35	41

Table 1. Number of facilities in the optimal solution of MW that coincide with users locations for the 287 users problem of Bongartz *et al.*, (1994).

In the limit for q going to infinity and a uniform distribution of sites solution of both problems coincide as any location in the plane is arbitrarily close to a site. But even for q moderate the relationship between both problems can be exploited. Quite often some of the locations chosen for a facility in the Multisource Weber problem coincide with users locations (see Table 1). Sufficient conditions for this to happen in the case $p = 1$, i.e., the usual Weber problem, are well-known (see e.g. Love, Morris and Wesolowsky (1988), Chapter 7). This suggests the following heuristic.

Heuristic MWPM (Multisource Weber as $p-$Median)

1. Define and solve a $p-$Median problem (PM) with the same users and demands as (MW) and the set of users locations of (MW) as potential sites for locating facilities. Let (t_i^*) (z_{ij}^*) be the optimal solution.

2. Let $I = \{i \mid t_i^* = 1\}$ and $C_i = \{j \mid z_{ij}^* = 1\}$ for $i \in I$. For each $i \in I$, solve the Weber problem with users set C_i. Let (x_i^*, y_i^*) be the optimal solution and f_i^* its value.

3. A heuristic solution for (MW) is given by $\{(x_i^*, y_i^*) \mid i \in I\}$ and (z_{ij}^*) with value $\sum_{i \in I} f_i^*$.

The solution obtained at completion of step 3 may be or not a local optimum for ALT. Since there is no guarantee for that (see Figure 1 for a counter-example), a Step 4 could be added to MWPM where either ALT or PROJ (methods whose convergence to a local optimum has been proved) would be applied to the solution obtained in Step 3.
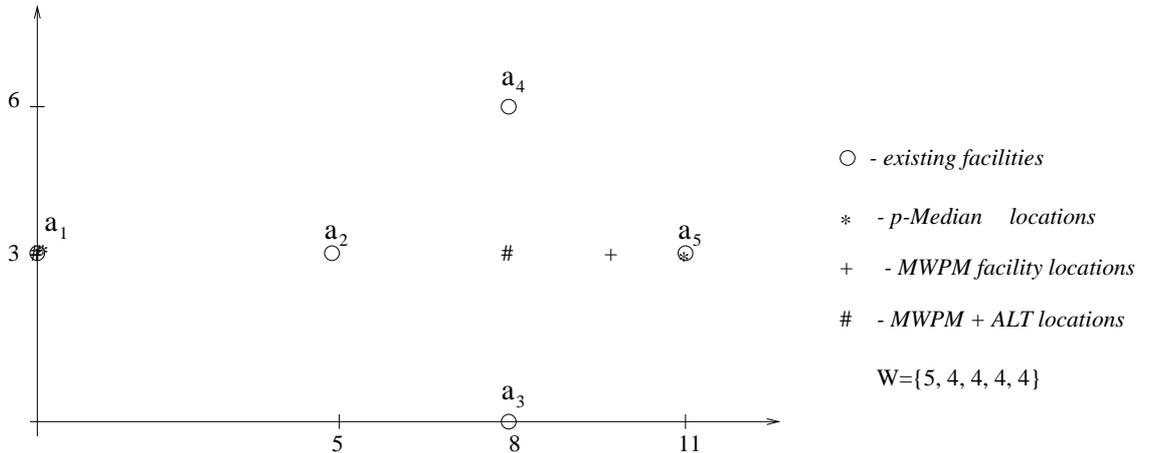
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Figure 1: Counter-example where reallocations improve the MWPM solution.

Computational experience reported in Table 4 shows this improves slightly the solution of MWPM on average.

We next construct an example to illustrate this improvement of the MWPM solution. Coordinates of $n = 5$ users are given as follows: $a_1 = (0, 3)$, $a_2 = (5, 3)$, $a_3 = (8, 0)$, $a_4 = (8, 6)$ and $a_5 = (11, 3)$. Their weights are $w_1 = 5$, $w_2 = 4$, $w_3 = 4$, $w_4 = 4$ and $w_5 = 4$. We want to locate $p = 2$ facilities. Optimal p-Median locations are at users locations a_1 and a_5 and the corresponding objective function value is $f_{PM} = 20 + 24\sqrt{2} = 53.9$ (Step 1 of MWPM). This solution defines a partition of the users set into two subsets: $\{a_1, a_2\}$ and $\{a_3, a_4, a_5\}$. In order to get the continuous solution, single facility Weber problems are solved for each users set (Step 2 of MWPM). Then the facilities have coordinates (0, 3) and (9.7, 3) with objective value $f_{MWPM} = 52.8$. However, if we continue with ALT, the user located at $a_2 = (5, 3)$ will change the facility he patronizes because now the point (9.7, 3) is closer to him than (0, 3). This reallocation followed by another location step leads to the optimal solution of the problem: (0,3) and (8,3) with $f_{opt} = 48$.

While the $p-$Median problem is NP-hard (Kariv and Hakimi (1969)), as is the Multisource Weber problem, it appears that the former is easier to solve exactly than the latter. We use for that purpose the very efficient code of Hanjoul and Peeters (1985). A simple code for Weiszfeld's algorithm is used for the less time consuming Step 2.

As mentioned above, heuristic MWPM is close to some ideas of Cooper (1963, 1964). Indeed, among several proposals, of which only ALT seems to have been explored in detail, Cooper (1963, 1964) suggests to consider in turn all $p-$tuples of users points as locations for the facilities. For each such $p-$tuple, users are assigned to the (or a) closest facility and the $p-$tuple for which the weighted sum of distances is smallest is kept. Weber problems are then solved for each facility's users set. As the number of $p-$tuples of users points may be very large, Cooper (1964) also suggests to draw a few
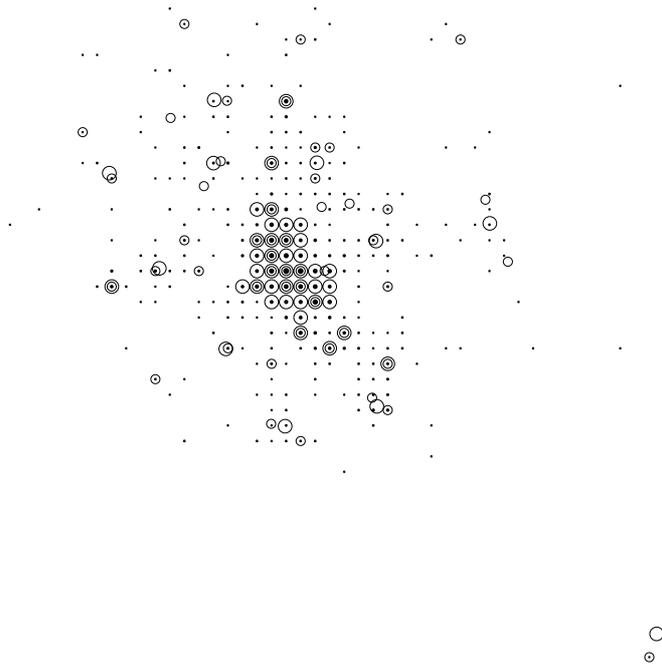
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Figure 2: 287 users problem with $p = 50$ facilities

of them at random and keep the best solution found. Heuristic MWPM makes Cooper's (1963, 1964) idea operational by identifying its first step as a $p-$median problem and using an efficient exact algorithm to solve it.

In the ALT heuristic p points are chosen as locations of the facilities, users assigned to a closest facility and Weber problems solved for each facility's set of users. Then the procedure is iterated with the new locations of the facilities until no more changes in assigments occur. In the multistart version (MALT) of this heuristic the procedure is repeated a given number of times and the best solution kept. Cooper (1963) (1964) suggests choosing initial points among users locations and we do the same. Bongartz *et al.* (1994) generate two different sets of starting points: TSP (traveling salesman problem) and random location starting points. We use only the first, faster option. In our first series of experiments the number of repetitions for MPROJ and for MALT is the same, i.e. $10p$ for $p \leq 10$ and 100 otherwise.

Results for two series of test problems are given in Table 2. The first series, due to Bongartz *et al.* (1994) have 287 users, corresponding to blocks of a city and $p = 2$ to 50 (not all values are reported to save space).[2] It appears that

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[2] There are two typos in the data of this problem as published in *Mathematical Programming* 66 (1994) page 308: user 39 located at (22, 25) has a weight of 1 and not 6; user 273 is located at (22, 45) and not at (22, 44). Data for this problem and the next one are also available at http://www.idsia.ch/~eric.

n	p	MWPM			MPROJ			MALT		
		f_{opt}	Time	%Error	f_{opt}	Time	%Error	f_{opt}	Time	%Error
287	2	14487.84	4.5	0.42	14427.59	11.1	0.00	14427.59	5.4	0.00
	3	12100.37	4.1	0.04	12095.44	24.3	0.00	12106.70	9.6	0.09
	4	10662.19	4.3	0.01	10661.48	44.5	0.00	10661.48	10.8	0.00
	5	9716.51	4.5	0.01	9715.63	64.7	0.00	9715.63	13.1	0.00
	6	8789.78	3.2	0.00	9082.11	88.3	3.33	8942.18	16.2	1.73
	7	8165.18	4.4	0.00	8337.81	117.0	2.11	8284.00	16.8	1.46
	8	7606.83	4.3	0.00	7796.16	151.0	2.49	7806.53	18.1	2.63
	9	7129.45	4.2	0.00	7401.01	200.2	3.81	7432.64	19.7	4.25
	10	6753.51	5.3	0.00	6861.23	249.4	1.60	7090.52	22.3	4.99
	11	6388.77	4.5	0.00	6797.95	273.3	6.40	6729.17	29.9	5.33
	12	6059.52	4.9	0.00	6381.19	309.0	5.31	6502.30	31.4	7.31
	13	5764.64	4.1	0.00	6456.26	315.2	12.00	6187.75	29.1	7.34
	14	5492.64	4.8	0.00	6120.85	359.8	11.44	6058.27	30.5	10.30
	15	5233.05	4.8	0.00	5977.99	389.6	14.24	5767.87	22.6	10.22
	20	4158.97	4.9	0.00	5061.83	553.4	21.71	5036.02	23.5	21.09
	25	3349.27	4.6	0.00	4453.77	695.9	32.98	4453.76	24.3	32.98
	30	2718.66	4.9	0.00	3989.97	819.0	46.76	3977.34	24.8	46.30
	35	2238.58	4.7	0.00	3506.79	925.6	56.65	3614.73	24.7	61.47
	40	1902.64	5.5	0.00	3380.22	1054.6	77.66	3367.19	25.7	76.97
	45	1632.57	4.4	0.00	3157.85	1219.2	93.43	3096.49	25.1	89.67
	50	1402.77	5.2	0.00	3025.35	1397.8	115.67	2854.02	26.9	103.46
Total error				0.48			507.57			487.58
654	2	815314.05	66.4	0.00	815313.30	28.7	0.00	815313.30	9.0	0.00
	3	552155.98	75.3	0.20	551062.88	54.9	0.00	551062.88	11.6	0.00
	4	288191.45	50.2	0.00	288190.99	93.1	0.00	288190.99	15.5	0.00
	5	209068.85	66.7	0.00	209068.79	151.1	0.00	209068.79	23.1	0.00
	6	180488.26	72.1	0.00	180488.21	259.9	0.00	180664.38	24.8	0.10
	7	163704.21	62.1	0.00	163704.17	295.1	0.00	163704.28	29.5	0.00
	8	147050.83	79.2	0.00	147050.79	408.9	0.00	147104.71	32.0	0.04
	9	130936.16	71.3	0.00	130936.12	526.8	0.00	131080.02	36.8	0.11
	10	115384.68	75.4	0.04	115339.03	641.7	0.00	116580.42	43.0	1.08
	11	100202.58	65.7	0.07	100133.20	763.8	0.00	107229.76	50.9	7.09
	12	94152.08	96.5	0.00	94152.05	699.4	0.00	97613.88	55.8	3.68
	13	89466.30	259.3	0.00	90850.39	704.1	1.55	90487.69	51.6	1.14
	14	84819.21	277.9	0.00	86233.92	623.9	1.67	86353.53	51.4	1.81
	15	80188.58	196.1	0.00	81584.70	366.5	1.74	82431.11	38.3	2.80
	20	63426.09	1357.9	0.00	68339.32	973.6	7.75	66384.60	42.1	4.66
	25	52274.98	1420.2	0.00	58640.01	1317.3	12.18	57945.67	46.8	10.85
	30	44802.81	4222.4	0.00	49962.54	1624.7	11.52	51474.53	52.0	14.89
	35	39404.87	1607.8	0.00	46825.08	1986.6	18.83	45534.34	56.9	15.56
	40	35904.71	1762.6	0.00	44698.22	2144.0	24.49	42355.87	60.2	17.97
	45	32467.50	1296.3	0.00	40755.25	2448.4	25.53	39584.92	64.6	21.92
	50	29462.98	2486.6	0.00	38925.55	2722.8	32.12	36321.78	71.1	23.28
Total error				0.31			137.36			126.96

Table 2. Computational results with heuristics MWPM, MPROJ and MALT.

(i) for $p = 2$ to 5, MPROJ and, with one exception, MALT give slightly better results than MWPM, the difference being very small except for the case $p = 2$;

(ii) for larger values of p MWPM gives better results than MPROJ and MALT and the magnitude of the difference steadily increases. For $p = 50$ the solution of MWPM (represented by large circles in *Figure 2* where dots represent demand points and their size increases with their weights) has a value *less than half* that of MPROJ (represented by small circles on that figure, and where facilities appear to be too dispersed);

(iii) computing times of MWPM are from 2 to 300 times smaller than those of MPROJ and from 1.2 to 7 times smaller than those of MALT.

The second series of test problems come from the Euclidean traveling salesman problem instance p654 of the TSPLIB library. They involve $n = 654$ users with unit weights and $2 \leq p \leq 50$. Results are in agreement with those of the first series except that exact solution of the $p-$median problem is now time consuming, hence computing times for MWPM and MPROJ are now comparable and larger than those of MALT.

	MWPM			MALT					
p	f_{opt}	*Time*	%Error	f_{opt}	*Time*	%Error	f_{opt}	*Time*	%Error
2	815314.06	66.4	0.00	815313.31	66.4	0.00	815313.31	664.4	0.00
3	552156.00	75.4	0.20	551062.88	75.4	0.00	551062.88	754.0	0.00
4	288191.44	50.3	0.00	288191.00	50.3	0.00	288191.00	503.4	0.00
5	209068.84	66.3	0.00	209068.80	66.3	0.00	209068.80	663.3	0.00
6	180488.27	72.4	0.00	180488.20	72.4	0.00	180488.20	724.1	0.00
7	163704.20	62.3	0.00	163704.17	62.3	0.00	163704.17	623.1	0.00
8	147050.83	79.6	0.00	147050.84	79.6	0.00	147050.80	795.7	0.00
9	130936.16	71.3	0.00	131046.12	71.3	0.08	130936.12	713.5	0.00
10	115384.68	75.3	0.02	115467.24	75.3	0.09	115357.72	752.5	0.00
11	100202.58	65.6	0.07	105194.92	65.6	5.05	100133.20	655.5	0.00
12	94152.08	96.3	0.00	95504.19	96.3	1.44	94152.05	963.0	0.00
13	89466.30	259.2	0.00	89946.19	259.2	0.54	89462.98	2591.8	0.00
14	84819.21	277.4	0.00	85007.74	277.4	0.22	84835.66	2773.7	0.02
15	80188.58	196.3	0.00	80903.62	196.3	0.89	80221.93	1962.6	0.04
20	63426.09	1357.2	0.02	64160.14	1357.2	1.18	63413.32	13572.0	0.00
25	52274.98	1420.3	0.00	55060.60	1420.3	5.33	53328.73	14203.3	2.02
30	44802.81	4222.2	0.00	48059.30	4222.2	7.27	47463.00	42222.4	5.94
35	39404.87	1607.2	0.00	43988.25	1607.2	11.63	43230.32	16072.1	9.71
40	35904.71	1762.2	0.00	39918.80	1762.2	11.18	38075.75	17622.5	6.05
45	32467.50	1296.4	0.00	36843.35	1296.4	13.48	36060.48	12963.6	11.07
50	29462.98	2486.3	0.00	34207.56	2486.3	16.10	33408.68	24863.1	13.39
Total error			0.32			74.49			48.23

Table 3. Computational results with heuristics MWPM and MALT on 654 users problem within the same and with 10 times more CPU time.

In the next table we compare MWPM and MALT on 654 users problem with the same CPU time as a stopping rule, since differences in CPU times between them (reported in Table 2) are large. It appears that the total % error for MALT reduces to 74.49%, which is still much larger than the 0.32% obtained by MWPM. The last 3 columns in Table 3 give results when the time allocated to MALT is 10 times that of MWPM. The

total % error drops to 48.23%, which is still large. This result indicates the presence of the so-called *central limit catastrophy* (Baum, 1985), already observed in multi start approaches to other combinatorial optimization problems (see Boese *et al.* for the traveling salesman and graph bisection problems and Hansen and Mladenović (1997) for the p-median problem).

		% DEVIATION						CPU TIME			
	Best		PM+	PM+	PM+				PM+		
p	found	PM	WSZ	ALT	PROJ	MPROJ	MALT	PM	PROJ	MPROJ	MALT
2	1923.73	0.07	0.03	0.00	0.00	0.00	0.00	5.9	5.9	25.8	5.4
3	1642.45	0.03	0.03	0.01	0.01	0.00	0.00	8.3	8.3	36.3	7.2
4	1442.66	0.15	0.04	0.02	0.00	0.00	0.00	10.6	10.6	58.2	10.2
5	1295.58	0.14	0.06	0.06	0.06	0.02	0.00	12.9	13.0	70.2	11.8
6	1185.62	0.09	0.04	0.03	0.03	0.03	0.00	14.9	15.0	72.8	13.9
7	1092.30	0.28	0.15	0.14	0.14	0.00	0.00	16.4	16.5	81.6	14.5
8	1012.70	0.10	0.01	0.01	0.01	0.01	0.00	25.8	25.9	103.2	18.0
9	953.44	0.24	0.09	0.01	0.01	0.00	0.00	23.7	23.8	118.3	19.3
10	904.27	0.23	0.03	0.00	0.00	0.13	0.00	23.1	23.2	133.2	18.3
11	858.22	0.27	0.12	0.03	0.03	0.01	0.00	26.4	26.5	150.0	19.8
12	820.20	0.46	0.28	0.13	0.13	1.62	0.00	26.7	26.8	148.7	21.7
13	783.85	0.39	0.08	0.00	0.00	0.17	0.31	29.3	29.5	169.0	21.3
14	755.35	0.37	0.17	0.00	0.00	1.46	0.96	30.6	30.8	173.4	22.3
15	725.57	0.63	0.17	0.00	0.00	0.60	1.74	28.5	28.7	191.7	23.6
20	616.48	0.28	0.08	0.00	0.00	1.59	4.11	38.1	38.5	217.2	32.3
25	535.86	0.37	0.02	0.00	0.00	3.01	7.45	43.6	43.9	313.8	39.3
30	478.24	0.63	0.13	0.00	0.00	3.95	9.03	48.8	49.3	365.5	45.3
35	434.43	0.46	0.03	0.00	0.00	4.53	12.03	55.7	56.2	460.7	49.4
40	396.32	0.73	0.25	0.00	0.00	6.39	14.63	57.0	57.6	529.0	50.5
45	365.21	0.55	0.11	0.00	0.00	8.18	19.29	57.1	57.9	587.0	49.6
50	337.60	0.51	0.11	0.00	0.00	10.34	19.61	64.6	65.3	658.0	48.4
Total error		7.00	2.00	0.44	0.42	42.03	89.17				

Table 4. Computational results for the 287 users problem with unit weights.

In Table 4 we compare again heuristics on the 287 users problem, but now with unit weights (as recommended by an anonymous referee). The exact p-median solution method of Hanjoul and Peeters (1985), becomes very time consuming for $p \geq 14$ (in fact, computing times jump from less than 20 seconds to about 2000 seconds). So, in order to show how close the solutions of PM and multisource Weber problems are, in Step 1 of MWPM we use the multistart *fast interchange* heuristic method of Whitaker (1985), Hansen and Mladenović (1997) instead of an exact algorithm. This heuristic is restarted 10 times and the best solution kept. In columns from 3 to 8 of Table 2 the % deviation relative to the best solution among methods compared are reported. In the last three columns CPU time of PM, PROJ and MALT are given. The CPU times of MWPM and PM+ALT are not reported because they are usually larger than that of PM by less than 0.05 sec. Thus, the running times of PM, MWPM and PM+ALT are almost the same. It appears that the integer solution obtained by the heuristic method (in Step 1 of MWPM) gives solutions of better quality than the state-of-the-art

heuristics MALT and MPROJ (compare the total 7% deviation for PM with 42% and 89% for MPROJ and MALT respectively). It appears also that the solution of PM are notably improved in Step 2, and a bit more if ALT or PROJ is used instead of the Weiszfeld (1937) procedure (which coincides with the first iteration of ALT).

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