A template for low-complexity implementation of POPMUSIC

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Abstract

POPMUSIC is a template for creating heuristics specially designed for dealing with large combinatorial problems that can be partially optimized. The basic idea is to optimize sub-parts of solutions until a local optimum is reached. The useful computational effort should be spent for optimizing these sub-parts. However, it is observed in practice that this computational effort grows almost linearly with problem size while the extraction of sub-parts has an higher algorithmic complexity. This work proposes a template for extracting sub-parts also in linear time.

1 Introduction

It is supposed in POPMUSIC template that a solution $s$ can be decomposed into $p$ parts $s_1, \ldots, s_q$ and that a proximity measure between two parts has been defined. Let $U$ be the set of parts that have not been used as seed-part for building a sub-problem. Initially, $U$ contains all $p$ parts and, at the end, $U$ is empty. Let $m$ be an optimization method that allows to optimize a sub-problem $R$ composed of $r$ parts.

With these hypothesis, the POPMUSIC template can be expressed by Algorithm 1

\begin{algorithm}
\caption{POPMUSIC template.}
\begin{algorithmic}
\Input Initial solution $s$ composed of $q$ parts $s_1, \ldots, s_q$, sub-problem optimization method $m$
\Output Improved solution $s$
\State $U = \{s_1, \ldots, s_q\}$
\While{$U \neq \emptyset$}
\State Select $s_g \in U$ // $s_g$: seed-part;
\State Build sub-problem $R$ with the $r$ parts of $s$ that are the closest to $s_g$;
\State Optimize $R$ with method $m$;
\If{$R$ improved}
\State Update solution $s$;
\State Remove from $U$ the parts that do not belong to $s$ anymore;
\State Insert in $U$ the parts of optimized sub-problem $R$;
\Else // $R$ not improved
\State Remove $s_g$ from $U$;
\EndIf
\EndWhile
\end{algorithmic}
\end{algorithm}

POPMUSIC has a unique parameter, $r$ which governs the size of sub-problems. This parameter is highly dependent on the procedure that is used for their optimization at line 5. When having chosen the sub-problem optimization procedure, it is generally relatively easy to find a good value for $r$. A very simple technique is to start with a small $r$ and increase its value until the computational time reaches a limit. This strategy is used for instance in [3].

Once the size of sub-problems is fixed, the computational effort for solving one of them can be considered as a constant. Since the while loop at line 2 is repeated a number of times that grows quasi-linearly with problem size [1, 2, 4, 5], it is very important for dealing with large problem instances to
limit the algorithmic complexity of the remaining steps of POPMUSIC, especially the generation of an
initial solution and step 4.

Building an initial solution with a good structure is essential for the success of a POPMUSIC imple-
mentation. Indeed, the template is supposed to be applied to problem instances meeting Special Intensi-
fication Conditions — explaining the last 3 letters of the acronym — where an element of the problem
is not supposed to have direct interactions with any other elements, but only with a limited number of
elements closely related. In the context of location-routing, [2] have proposed to build an initial solution
on the base of a p-median solution. However, building an initial solution is highly dependent on the
problem under consideration and it is not possible to provide a general template for having a procedure
with low algorithmic complexity.

In the present work, we suggest to use also a p-median solution for building closeness relations
among the \( n \) elements of a problem instance in \( O(n^{3/2}) \).

2 Creating closeness relations

For creating closeness relations, we make hypothesis that are commonly met when dealing with large
problem instances:

- A closeness function \( d(i,j) \) that can be computed independently from the problem size is available
  for measuring the distance between any pair \((i,j)\) of elements of the problem.
- The maximal number of elements of a sub-problem is fixed and does not depend on the total
  number of elements of the problem.
- Relatively few elements (at most \( \sqrt{n} \) on the average) can be considered as close to a given element.

Without a closeness data structure built in a preprocessing phase, a POPMUSIC implementation may
have a complexity growing at least quadratically, which is impracticable for large instances. In such a
case, instead of spending most computational effort for performing useful work (the optimization of
sub-problems), most effort is spent for building sub-problems.

The present work suggests to build the closeness relationship by first solving a kind of p-median
problem on the entities of the problem, by choosing \( p = \sqrt{n} \). The pure p-median problem aims at
choosing \( p \) central element for creating \( p \) clusters, where each element is assigned its closest centre and
with the objective to minimize the sum of the distance between elements and centres.

However, the p-median problem is NP-hard and an optimal solution is not necessarily the best for
creating a closeness relation. Indeed, few clusters might contain most of the elements, if the data have a
very dense region, while others clusters might contains only few elements. So, the number of elements
must be balanced and contain about \( p/n = \sqrt{n} \) elements each. To get such a clustering, we suggest
to use a variation of the method proposed in [2] for creating an initial solution to large location-routing
problem. Algorithm 2 presents how to solve heuristically this p-median problem with soft capacity.
When invoked with \( p = \sqrt{n} \), Algorithm 2 has a complexity in \( O(n^{3/2}) \), due to step 14.

3 Computational results

A suggested value for parameter \( \text{sample size} \) is 30, which speed-up a lot the computational time when \( n \)
is large and allows to produce relatively good solutions. With this parameters and performing \( \text{iter} = 30 \)
iterations, the computational time is limited to few seconds for problem instances up to 100,000 entities.

Even if it tries to solve a problem more complex than the pure p-median, this algorithm is generally
able to find better solutions than a k-means-like heuristics (an improving method that alternate allocation
and centres positioning until all entities are allocated to the best possible centre and all centres are at the
best possible place among the entities allocated to them), as shown in Table 1.
Algorithm 2: Generating a p-median with well balanced clusters

**Input:** n entities, number p of clusters, closeness measure \( d(i,j) \) between entities, parameters sample size and iter

**Output:** p clusters

1. Randomly select a sample of \( e = \min(n, \text{sample size} \cdot p) \) elements;
2. Randomly choose \( p \) centres among the \( e \) entities;
3. \( f = 0.6; \lambda_j = 0; j = 1 \ldots p; \)
4. **for** iter times **do**
   5. **for** \( i = 1 \) to \( e \) **do**
      6. Allocate entity \( i \) to the centre \( j \) minimizing \( d(i,j) + \lambda_j \)
   7. **for Each of the p clusters do**
      8. Find the best position of the centre among entities assigned to that centre
      9. Compute solutions cost \( C \) and store solution if it improves best solution found;
   10. \( f \leftarrow 0.99 \cdot f; \)
   11. **for** \( j = 1 \) to \( p \) **do**
      12. Compute the number \( n_j \) of entities allocated to centre \( j; \)
         \( \lambda_j \leftarrow \max (0, \lambda_j + f \cdot C \cdot (n_j - e/p)/e^2) \);
   14. Allocate all \( n \) entities to the best position found for the \( p \) centres;

<table>
<thead>
<tr>
<th>Instance</th>
<th>size</th>
<th>Sample cost</th>
<th>Final allocation</th>
<th>Cluster imbalance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finland</td>
<td>10639</td>
<td>-4.9</td>
<td>-5.2</td>
<td>-36.7</td>
</tr>
<tr>
<td>Italy</td>
<td>16862</td>
<td>-2.4</td>
<td>-3.8</td>
<td>-36.0</td>
</tr>
<tr>
<td>Sweden</td>
<td>24978</td>
<td>-2.9</td>
<td>-4.5</td>
<td>-42.1</td>
</tr>
<tr>
<td>China</td>
<td>71009</td>
<td>-3.5</td>
<td>-4.7</td>
<td>-29.3</td>
</tr>
</tbody>
</table>

Table 1: Improvement of performances of Algorithm 2 over k-means like heuristics, expressed in percent. Problem instances are issued from National TSP library.

Once \( p = \sqrt{n} \) centres positions have been found, several possibilities exist for defining closeness relations between centres while using methods easy to implement. A first possibility is to retain for each centre the \( r \) closest ones by complete enumeration. Two entities allocated to the same centre or to centres that are close might be close and the closeness relation can be computed between these entities. The complexity of building relations between entities with that technique is \( O(n^{3/2}) \).

Another way is to consider that two centres are close if one if the closest of an entity and the other the second closest one. Figure 1 illustrates this way of defining closeness relations.

References


(a) Problem decomposition into clusters  
(b) Building proximity network

Figure 1: Illustration of the decomposition of a problem into $\sqrt{n}$ clusters and building a proximity network between the clusters.